Control Systems on the Engel Group: Equivalence and Classification

Presentation · July 2016
DOI: 10.13140/RG.2.1.4997.0161

2 authors:

Catherine Eve Mc Lean. Née Bartlett
Rhodes University
6 PUBLICATIONS  7 CITATIONS

Claudiu Cristian Remsing
Rhodes University
66 PUBLICATIONS  303 CITATIONS

Some of the authors of this publication are also working on these related projects:

- Quadratic Hamilton-Poisson Systems View project
- Invariant Control Systems on Lie Groups View project
Control Systems on the Engel Group

Equivalence and Classification

C.E. Bartlett      C.C. Remsing*

Geometry, Graphs and Control (GGC) Research Group
Department of Mathematics, Rhodes University
6140 Grahamstown, South Africa

Differential Geometry and its Applications
Brno, Czech Republic, July 11-15, 2016
Outline

- The Engel group
- Invariant control systems
  - DF-equivalence
  - Classification under DF-equivalence
  - SDF-equivalence
  - Classification under SDF-equivalence
- Invariant optimal control
  - Cost-extended control systems
  - C-equivalence
  - Classification under C-equivalence
The Engel group $\text{Eng}$

Matrix representation

$$\text{Eng} = \left\{ \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & w \\ 0 & 1 & z & z-x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$$

Fact

$\text{Eng}$ is a 4D (connected and simply connected) nilpotent Lie group.
The Engel Lie algebra \( \mathfrak{eng} \)

### Matrix representation

\[
\mathfrak{eng} = \begin{cases} 
\begin{bmatrix}
0 & z & 0 & w \\
0 & 0 & z & x \\
0 & 0 & 0 & y \\
0 & 0 & 0 & 0 \\
\end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 : w, x, y, z \in \mathbb{R} 
\end{cases}
\]
Invariant control systems

Left-invariant control affine system

\[ \Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell \]

\[ \frac{dg}{dt} = \Xi(g, u) = g (A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \text{Eng}, \ u \in \mathbb{R}^\ell \]

- **state space**: the Engel group \( \text{Eng} \) (a connected Lie group, in general)
- **input set**: \( \mathbb{R}^\ell, \ 1 \leq \ell \leq 4 \)
- \( A, B_1, \ldots, B_\ell \in \text{eng} \)
- **dynamics**: family of left-invariant vector fields \( \Xi_u = \Xi(\cdot, u) \)
- **parametrization map**: \( \Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \rightarrow \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell \) is an injective (affine) map
admissible controls: piecewise continuous curves $u(\cdot) : [0, T] \rightarrow \mathbb{R}^\ell$

trajectory: absolutely continuous curve s.t. $\dot{g}(t) = \Xi(g(t), u(t))$

controlled trajectory: pair $(g(\cdot), u(\cdot))$

controllable system: any two states can be joined by a trajectory

full rank: $\text{Lie}(\Gamma) = \mathfrak{g}$ (necessary condition for controllability)

$\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell$

trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$

homogeneous: $A \in \Gamma^0$

inhomogeneous: $A \notin \Gamma^0$
Control system

\[
\frac{dg}{dt} = \Xi(g, u) = g(A + u_1B_1 + \cdots + u_\ell B_\ell)
\]

Definition (DF-equivalence)

\(\Sigma\) and \(\Sigma'\) are DF-equivalent if there exist

- a diffeomorphism \(\phi : \text{Eng} \to \text{Eng}\)
- an affine isomorphism \(\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell\)

such that

\[
T_{g\phi} \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad (g \in \text{Eng}, u \in \mathbb{R}^\ell).
\]
Remark (DF-equivalence)

- one-to-one correspondence between trajectories
- $\phi$ preserves left-invariant vector fields:

$$\phi_*\Xi_u = \Xi'_\varphi(u) \quad (\Xi_u = \Xi(\cdot, u))$$

Characterization (for simply connected Lie groups, in general)

Full-rank systems $\Sigma$ and $\Sigma'$ are DF-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \text{Aut}(\mathfrak{e}ng)$ such that

$$\psi \cdot \Gamma = \Gamma'.$$
Single-input (inhomogeneous) control system

\[ \Sigma : \ A + uB \]

Classification (1,1)

Any full-rank single-input control system is DF-equivalent to exactly one of the following systems

\[ \Sigma_1^{(1,1)} : \ E_3 + uE_4 \]
\[ \Sigma_2^{(1,1)} : \ E_4 + uE_3 \]
Classification (under DF-equivalence): two inputs

Two-input control system

\[ \Sigma : \quad A + u_1 B_1 + u_2 B_2 \]

Classification (2,0)

Any full-rank two-input homogeneous control system is DF-equivalent to the system

\[ \Sigma^{(2,0)} : \quad u_1 E_3 + u_2 E_4. \]
Classification (under DF-equivalence): two inputs

Any full-rank two-input inhomogeneous control system is DF-equivalent to exactly one of the following systems:

\[
\Sigma^{(2,1)}_1 : \quad E_4 + u_1 E_1 + u_2 E_3 \quad \Sigma^{(2,1)}_2 : \quad E_3 + u_1 E_1 + u_2 E_4
\]

\[
\Sigma^{(2,1)}_3 : \quad E_4 + u_1 E_2 + u_2 E_3 \quad \Sigma^{(2,1)}_4 : \quad E_3 + u_1 E_2 + u_2 E_4
\]

\[
\Sigma^{(2,1)}_5 : \quad E_1 + u_1 E_3 + u_2 E_4 \quad \Sigma^{(2,1)}_6 : \quad E_2 - E_1 + u_1 E_3 + u_2 E_4
\]

\[
\Sigma^{(2,1)}_7 : \quad E_2 + u_1 E_3 + u_2 E_4.
\]
Classification (under DF-equivalence): three inputs

Three-input control system

\[ \Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification (3,0)

Any full-rank three-input homogeneous control system is DF-equivalent to exactly one of the systems

\[ \Sigma^{(3,0)}_1 : \quad u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(3,0)}_2 : \quad u_1 E_2 + u_2 E_3 + u_3 E_4. \]
Classification (under DF-equivalence): three inputs

Three-input control system

\[ \Sigma : \ A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification (3,1)

Any full-rank three-input inhomogeneous control system is DF-equivalent to exactly one of the following systems

\[ \Sigma_{1}^{(3,1)} : E_3 + u_1 E_1 + u_2 E_2 + u_3 E_4 \]
\[ \Sigma_{2}^{(3,1)} : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma_{3}^{(3,1)} : E_1 + u_1 E_2 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma_{4}^{(3,1)} : E_4 + u_1 E_1 + u_2 E_2 + u_3 E_3. \]
Control system

\[
\begin{align*}
(\Sigma) & \quad \frac{dg}{dt} = \Xi(g, u) \\
& = g (A + u_1 B_1 + \cdots + u_\ell B_\ell)
\end{align*}
\]

Definition (SDF-equivalence)

\[\Sigma\] and \(\Sigma'\) are SDF-equivalent if there exist

- a diffeomorphism \(\phi : \text{Eng} \rightarrow \text{Eng}\)
- a linear isomorphism \(\varphi : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell\)

such that

\[T_g \phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad (g \in \text{Eng}, u \in \mathbb{R}^\ell).\]
Remark (SDF-equivalence)

- one-to-one correspondence between trajectories
- $\phi$ preserves left-invariant vector fields:

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \quad (\Xi_u = \Xi(\cdot, u))$$

Characterization (for simply connected Lie groups, in general)

Full-rank systems $\Sigma$ and $\Sigma'$ are SDF-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \text{Aut}(\mathfrak{eng})$ such that

$$\psi \cdot \Gamma = \Gamma' \quad \text{and} \quad \psi \cdot A = A'.$$
Classification (under SDF-equivalence): single input

Single-input (inhomogeneous) control system

\[ \Sigma : A + uB. \]

Classification (1,1)

(a) If \( \Sigma \) is DF-equivalent to \( \Sigma_1^{(1,1)} : E_3 + uE_4 \), then \( \Sigma \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma_1^{(1,1)} : E_3 + uE_4 \quad \Sigma_{12}^{(1,1)} : E_3 + E_4 + uE_4. \]

(b) If \( \Sigma \) is DF-equivalent to \( \Sigma_2^{(1,1)} \), then \( \Sigma \) is SDF-equivalent to \( \Sigma_2^{(1,1)} \).
Classification (under SDF-equivalence): two inputs

Two-input control system

\[ \Sigma : A + u_1 B_1 + u_2 B_2 \]

Classification (2,0)

The system \( \Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \]

\[ \Sigma^{(2,0)}_{12} : E_3 + u_1 E_3 + u_2 E_4 \]

\[ \Sigma^{(2,0)}_{13} : E_4 + u_1 E_3 + u_2 E_4 . \]
Two-input control system

\[ \Sigma : \quad A + u_1 B_1 + u_2 B_2 \]

Classification: 1/7

The system \( \Sigma_{4}^{(2,1)} : E_3 + u_1 E_2 + u_2 E_4 \) is **SDF-equivalent** to exactly one of the following systems

\[ \Sigma_{4}^{(2,1)} : \quad E_3 + u_1 E_2 + u_2 E_4 \quad \Sigma_{42}^{(2,1)} : \quad E_2 + E_3 + u_1 E_2 + u_2 E_4 \]

\[ \Sigma_{43}^{(2,1)} : \quad E_3 + E_4 + u_1 E_2 + u_2 E_4. \]
Classification (under SDF-equivalence): three inputs

Three-input control system

\[ \Sigma : A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification: 1/2

The system \( \Sigma^{(3,0)}_1 : u_1 E_1 + u_2 E_3 + u_3 E_4 \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma^{(3,0)}_1 : u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(3,0)}_{12} : E_1 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(3,0)}_{13} : E_3 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(3,0)}_{14} : E_4 + u_1 E_1 + u_2 E_3 + u_3 E_4. \]
**Classification (under SDF-equivalence): three inputs**

**Three-input control system**

\[ \Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

**Classification: 1/4 (3,1)**

The system \( \Sigma^{(3,1)}_2 : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \) is \textit{SDF-equivalent} to exactly one of the following systems:

\[ \Sigma^{(3,1)}_2 : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]

\[ \Sigma^{(3,1)}_{22} : E_2 + E_3 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]

\[ \Sigma^{(3,0)}_{23} : E_2 + E_4 + u_1 E_1 + u_2 E_3 + u_3 E_4. \]
Invariant optimal control problems

Problem

Minimize cost functional $J = \int_0^T \chi(u(t)) \, dt$
over controlled trajectories of a system $\Sigma$
subject to boundary data.

Formal statement

LiCP

$$\frac{dg}{dt} = g\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right), \quad g \in \text{Eng}, \ u \in \mathbb{R}^\ell$$

$$g(0) = g_0, \ g(T) = g_1$$

$$J = \int_0^T (u(t) - \mu)^\top Q (u(t) - \mu) \, dt \rightarrow \min$$

$(\mu \in \mathbb{R}^\ell, \ Q \in \mathbb{R}^{\ell \times \ell} \text{ positive definite})$
Cost-extended control system

Definition

A **cost-extended control system** is a pair \((\Sigma, \chi)\), where

- \(\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell\) (left-invariant control affine system)
- \(\chi(u) = (u(t) - \mu)^\top Q (u(t) - \mu)\) (quadratic cost).

**Remark**

\((\Sigma, \chi) + \text{boundary data} \iff \text{invariant optimal control problem}\)
Cost equivalence

Cost-extended control system

- $\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell$
- $\chi(u) = (u(t) - \mu)\top Q (u(t) - \mu)$

Definition (C-equivalence)

$(\Sigma, \chi)$ and $(\Sigma', \chi')$ are C-equivalent if there exist

- a Lie group isomorphism $\phi : \text{Eng} \to \text{Eng}$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$\phi_* \Xi_u = \Xi'_{\varphi(u)}$ and $\exists r > 0 \quad \chi' \circ \varphi = r \chi.$
Remark 1

\((\Sigma, \chi)\) and \((\Sigma', \chi')\)

\(\Rightarrow\)

\(\Sigma\) and \(\Sigma'\)

cost equivalent

detached feedback equivalent

Remark 2

\(\Sigma\) and \(\Sigma'\)

detached feedback equivalent

w.r.t. \(\varphi \in \text{Aff}(\mathbb{R}^\ell)\)

\(\Rightarrow\)

\((\Sigma, \chi \circ \varphi)\) and \((\Sigma', \chi)\)

cost equivalent for any \(\chi\)
Classification under C-equivalence: algorithm

1. classify underlying systems under DF-equivalence
2. for each normal form $\Sigma_i$,
   - determine transformations $T_{\Sigma_i}$ preserving system $\Sigma_i$
   - normalize (admissible) associated cost $\chi$ by dilating by $r > 0$ and composing with $\varphi \in T_{\Sigma_i}$

\[
T_{\Sigma} = \left\{ \varphi \in \text{Aff} (\mathbb{R}^{\ell}) : \exists \psi \in d \text{Aut}(\text{Eng}), \psi \cdot \Gamma = \Gamma \\
\quad \quad \quad \quad \quad \psi \cdot \Xi(1, u) = \Xi(1, \varphi(u)) \right\}
\]
Controllability of systems (on the Engel group)  (1/2)

Definition (general)
A control system $\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell$ is controllable if any two states can be joined by a trajectory.

Remark 1
Any controllable system has full rank (i.e., $\text{Lie}(\Gamma) = \text{eng}$).

Remark 2
Any nilpotent Lie group is completely solvable.

Controllability criterion (for simply connected completely solvable Lie groups, in general)
A control system (on $\text{Eng}$) is controllable if and only if $\text{Lie}(\Gamma^0) = \text{eng}$. 
Controllability of systems (on the Engel group)  (2/2)

\[ \Sigma : \quad A + u_1 B_1 + \cdots + u_\ell B_\ell \quad (A, B_1, \ldots, B_\ell \in \text{eng}, \ 1 \leq \ell \leq 4) \]

Result

Any \textit{controllable} control system \( \Sigma \) is DF-equivalent to exactly one of the following (nine) systems

- \( \Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \)
- \( \Sigma^{(2,1)}_5 : E_1 + u_1 E_3 + u_2 E_4 \)
- \( \Sigma^{(2,1)}_6 : E_2 - E_1 + u_1 E_3 + u_2 E_4 \)
- \( \Sigma^{(2,1)}_7 : E_2 + u_1 E_3 + u_2 E_4 \)
- \( \Sigma^{(3,0)}_1 : u_1 E_1 + u_2 E_3 + u_3 E_4 \)
- \( \Sigma^{(3,0)}_2 : u_1 E_2 + u_2 E_3 + u_3 E_4 \)
- \( \Sigma^{(3,1)}_2 : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \)
- \( \Sigma^{(3,1)}_3 : E_1 + u_1 E_1 + u_2 E_3 + u_3 E_4 \)
- \( \Sigma^{(4,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4 \).
Any controllable cost-extended system \((\Sigma, \chi)\) with \(\Sigma\) DF-equivalent to 
\[
\Sigma^{(2,0)} : u_1 E_3 + u_2 E_4
\]
is C-equivalent to exactly one of the following systems

\[
(\Sigma^{(2,0)}, \chi^{(2,0)}) : \begin{cases} 
\Sigma^{(2,0)} & : u_1 E_3 + u_2 E_4 \\
\chi(u) = u_1^2 + u_2^2
\end{cases}
\]

\[
(\Sigma^{(2,0)}_{12}, \chi^{(2,0)}) : \begin{cases} 
\Sigma^{(2,0)}_{12} & : E_3 + u_1 E_3 + u_2 E_4 \\
\chi(u) = u_1^2 + u_2^2
\end{cases}
\]

\[
(\Sigma^{(2,0)}_{13\beta}, \chi^{(2,0)}) : \begin{cases} 
\Sigma^{(2,0)}_{13\beta} & : E_4 + \beta E_3 + u_1 E_3 + u_2 E_4 \\
\chi(u) = u_1^2 + u_2^2
\end{cases}
\]

Here \(\beta \geq 0\) parametrizes a family of distinct class representatives.