Nonholonomic Riemannian Structures on Lie Groups

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Nonholonomic Riemannian Structures on Lie Groups

Equivalence • Classification • Flat Structures

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Introduction

Nonholonomic Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$

Model for motion of free particle

- moving in configuration space M with kinetic energy Lagrangian
- constrained to move in "admissible directions" $\mathcal{D}$

Invariant structures on Lie groups are of the most interest

Objective

Primary

- classify all invariant structures on 3D Lie groups
- describe equivalence classes in terms of scalar invariants

Secondary

- classify the invariant 3D flat structures

Outline

Nonholonomic Riemannian structure $(\mathrm{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$

Ingredients

- M is an n-dim manifold
- $\mathcal{D}$ is a completely nonholonomic, rank $r < n$ distribution on M
- $TM = \mathcal{D} \oplus \mathcal{D}^\perp$, with projectors $\mathscr{P} : TM \to \mathcal{D}$ and $\mathscr{Q} : TM \to \mathcal{D}^\perp$
- $\mathbf{g}$ is a fibre metric on $\mathcal{D}$

NH connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

unique connection such that

$$\nabla \mathbf{g} \equiv 0 \quad \text{and} \quad \nabla_X Y - \nabla_Y X = \mathscr{P}([X, Y])$$

- parallel transport only along integral curves of $\mathcal{D}$
- $\mathcal{D}$-curve γ is a nonholonomic geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$

Nonholonomic isometries

NH-isometry $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g}) \to (\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$

diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}', \quad \phi_* \mathcal{D}^\perp = \mathcal{D}'^\perp \quad \text{and} \quad \mathbf{g} = \phi^* \mathbf{g}'$$

Properties

- preserves the nonholonomic connection: $\nabla = \phi^* \nabla'$
- establishes a 1-to-1 correspondence between the nonholonomic geodesics of the two structures
- preserves the projectors: $\phi_* \mathscr{P}(X) = \mathscr{P}'(\phi_* X)$ for every $X \in \Gamma(T\mathsf{M})$

Left-invariant nonholonomic Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$

- $\mathsf{M} = \mathsf{G}$ is a Lie group
- left translations $L_g : h \mapsto gh$ are NH-isometries

Nonholonomic Riemannian structures in 3D

Contact structure on M

We have $\mathcal{D} = \ker \omega$, where ω is a 1-form on M such that

$$\omega \wedge d\omega \neq 0$$

- fixed up to sign by condition:

$$|d\omega(Y_1, Y_2)| = 1, \qquad (Y_1, Y_2) \text{ o.n. frame for } \mathcal{D}$$

- Reeb vector field $Y_0 \in \Gamma(TM)$:

$$i_{Y_0}\omega = 1 \qquad \text{and} \qquad i_{Y_0} d\omega = 0$$

Isometric invariants

- $\vartheta = \|\mathscr{P}(Y_0)\|^2 \geq 0$
- κ, χ_1, χ_2 — curvature invariants
- two natural cases: $\vartheta = 0$ and $\vartheta > 0$

Bianchi–Behr classification of 3D Lie algebras

Unimodular algebras and (simply connected) groups

Lie algebra	Lie group	Name	Class
$\mathbb{R}^3$	$\mathbb{R}^3$	Abelian	Abelian
$\mathfrak{h}_3$	H_3	Heisenberg	nilpotent
$\mathfrak{se}(1,1)$	$\mathsf{SE}(1,1)$	semi-Euclidean	completely solvable
$\mathfrak{se}(2)$	$\widetilde{\mathsf{SE}}(2)$	Euclidean	solvable
$\mathfrak{sl}(2,\mathbb{R})$	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$	special linear	semisimple
$\mathfrak{su}(2)$	$\mathsf{SU}(2)$	special unitary	semisimple

Non-unimodular (simply connected) groups

$$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \quad \mathsf{G}_{3.2}, \quad \mathsf{G}_{3.3}, \quad \mathsf{G}_{3.4}^h \; (h > 0,\; h \neq 1), \quad \mathsf{G}_{3.5}^h \; (h > 0)$$

Left-invariant distributions on 3D groups

Killing form

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad \mathcal{K}(U, V) = \mathrm{tr}[U, [V, \cdot]]$$

- $\mathcal{K}$ is nondegenerate $\iff$ $\mathfrak{g}$ is semisimple

Completely nonholonomic left-invariant distributions on 3D groups

- no such distributions on $\mathbb{R}^3$ or $\mathsf{G}_{3.3}$

Up to Lie group automorphism:

- exactly one distribution on H_3, $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$, $\mathsf{SU}(2)$ and non-unimodular groups
- exactly two distributions on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$:

denote	$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$	if $\mathcal{K}$ indefinite on $\mathcal{D}$
"	$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$	" " definite " "

Case 1: $\vartheta = 0$

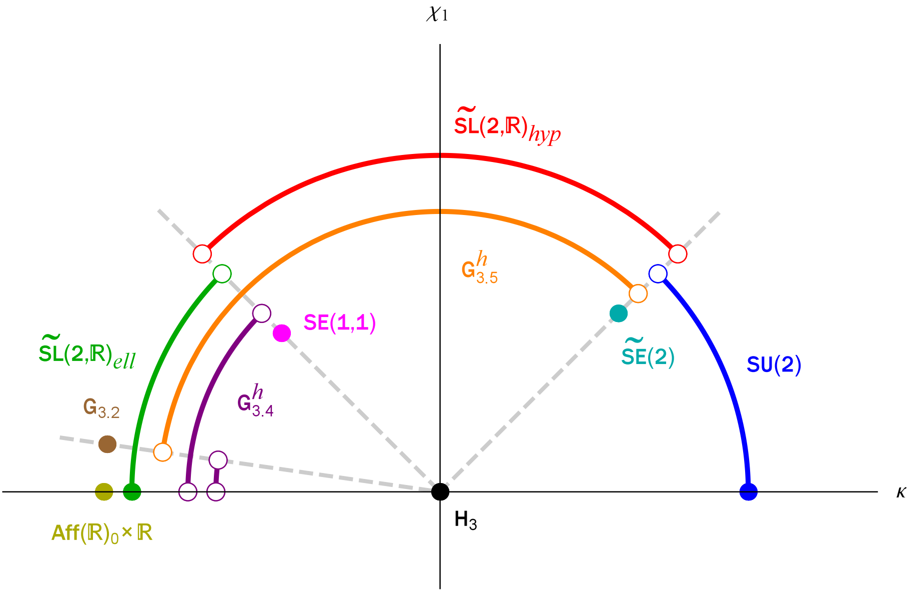
- $\mathcal{D}^{\perp} = \text{span}\{Y_0\}$ determined by $(\mathcal{D}, \mathbf{g})$
- reduces to a sub-Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathbf{g})$
- invariant sub-Riemannian structures classified by Agrachev & Barilari (2012)

Invariants

- $\{\kappa, \chi_1\}$ form a complete set of invariants for structures on unimodular groups
- structures on non-unimodular groups are further distinguished by discrete invariants
- can rescale structures so that

$$\kappa = \chi_1 = 0 \qquad \text{or} \qquad \kappa^2 + \chi_1^2 = 1$$

Classification when $\vartheta = 0$



Case 2: $\vartheta > 0$

Canonical frame (X_0, X_1, X_2)

$$X_0 = \mathscr{Q}(Y_0) \qquad X_1 = \frac{\mathscr{P}(Y_0)}{\|\mathscr{P}(Y_0)\|} \qquad X_2 \text{ unique unit vector s.t. } d\omega(X_1, X_2) = 1$$

- $\mathcal{D} = \operatorname{span}\{X_1, X_2\}$, $\mathcal{D}^\perp = \operatorname{span}\{X_0\}$
- canonical frame (up to sign of X_0, X_1) on M

Commutator relations (determine structure uniquely)

$$\begin{cases} [X_1, X_0] = c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2, X_0] = c_{20}^0 X_0 + c_{20}^1 X_1 + c_{20}^2 X_2 \\ [X_2, X_1] = X_0 + c_{21}^1 X_1 + c_{21}^2 X_2 \end{cases} \qquad c_{ij}^k \in \mathcal{C}^\infty(\mathrm{M})$$

Left-invariant structures

- canonical frame (X_0, X_1, X_2) is left invariant
- ϑ, κ, χ_1, χ_2 and c_{ij}^k are constant

NH-isometries preserve the Lie group structure

$(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ NH-isometric to $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ w.r.t. $\phi : \mathsf{G} \to \mathsf{G}'$ $\Longrightarrow$ $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where ϕ' is a Lie group isomorphism

- hence NH-isometries preserve the Killing form $\mathcal{K}$

Three new invariants ϱ_0, ϱ_1, ϱ_2

$$\varrho_i = -\tfrac{1}{2}\mathcal{K}(X_i, X_i), \qquad i = 0, 1, 2$$

Classification

Approach

- rescale frame so that $\vartheta = 1$
- split into cases depending on structure constants
- determine group from commutator relations

Example: G is unimodular and $c_{10}^1 = c_{10}^2 = 0$

$$[X_1, X_0] = 0 \qquad [X_2, X_0] = -X_0 + c_{20}^1 X_1 \qquad [X_2, X_1] = X_0 + X_1$$

- implies $\mathcal{K}$ is degenerate (i.e., G not semisimple)

(1)	$c_{20}^1 + 1 > 0$	$\Longrightarrow$	compl. solvable	hence on SE(1, 1)
(2)	$c_{20}^1 + 1 = 0$	$\Longrightarrow$	nilpotent	" " H_3
(3)	$c_{20}^1 + 1 < 0$	$\Longrightarrow$	solvable	" " $\widetilde{\mathrm{SE}}(2)$

- for SE(1, 1), $\widetilde{\mathrm{SE}}(2)$: c_{20}^1 is a parameter (i.e., family of equiv. classes)

Selected results (solvable groups)

H_3

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - X_1 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = 0 \end{cases}$$

$\mathsf{SE}(1,1)$

$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\, X_1 - \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1-\alpha_2) X_1 + \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\alpha_1 \\ \varrho_1 = -\alpha_2 \\ \varrho_2 = -\alpha_2 \end{cases}$$

$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

$\widetilde{\mathsf{SE}}(2)$

$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\, X_1 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1+\alpha_2) X_1 + \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = \alpha_1 \\ \varrho_1 = \alpha_2 \\ \varrho_2 = \alpha_2 \end{cases}$$

$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

Remarks

Structures on unimodular groups

- $\{\vartheta, \varrho_0, \varrho_1, \varrho_2\}$ form a complete set of invariants
- $\{\vartheta, \kappa, \chi_1\}$ also suffice for H_3, $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$
- $\chi_2 = 0$

Structures on 3D non-unimodular groups

On a fixed non-unimodular Lie group (except for $\mathsf{G}_{3.5}^1$), there exist at most two non-NH-isometric structures with the same invariants ϑ, ϱ_0, ϱ_1, ϱ_2

- exception $\mathsf{G}_{3.5}^1$: infinitely many ($\varrho_0 = \varrho_1 = \varrho_2 = 0$)
- use κ, χ_1 or χ_2 to form complete set of invariants

Flat structures

Parallel frames

- an o.n. frame (X_a) for $\mathcal{D}$ is parallel if $\nabla X_a \equiv 0$
- associated to ∇ is the parallel translation map along a $\mathcal{D}$-curve γ:

$$\Pi_\gamma^t : \mathcal{D}_{\gamma(0)} \to \mathcal{D}_{\gamma(t)}$$

- Π_γ^t is independent of $\gamma \iff$ there exists a parallel frame for $\mathcal{D}$

$(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is called flat if there exists a parallel frame for $\mathcal{D}$

Objective

- classify flat structures in three dimensions

Selected results (unimodular groups)

$\vartheta = 0$

- all structures are flat

$\vartheta > 0$

- flat iff. $\varrho_0 = \vartheta \varrho_1$ and $\varrho_2 = 0$
    - $\varrho_0 < 0 \quad \Longrightarrow \quad$ 1-parameter family of structures on $\mathsf{SE}(1,1)$
    - $\varrho_0 = 0 \quad \Longrightarrow \quad$ structure on H_3
    - $\varrho_0 > 0 \quad \Longrightarrow \quad$ 1-parameter family of structures on $\widetilde{\mathsf{SE}}(2)$

Interesting observations

- every structure on H_3 and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is flat
- the only flat structures on the semisimple groups are "trivial" structures (those whose NH geodesics are 1-parameter subgroups)
- the remaining groups all admit a 1-parameter family of flat structures (up to equivalence)

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