Nonholonomic Riemannian Structures on Lie Groups

Equivalence • Classification • Flat Structures

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Conference on Geometry: Theory and Applications
Pilsen, Czech Republic, June 26–30, 2017
Introduction

Nonholonomic Riemannian structure \((M, D, D^\perp, \mathbf{g})\)

Model for motion of free particle

- moving in configuration space \(M\) with kinetic energy Lagrangian
- constrained to move in “admissible directions” \(D\)

Invariant structures on Lie groups are of the most interest

Objective

Primary

- classify all invariant structures on 3D Lie groups
- describe equivalence classes in terms of scalar invariants

Secondary

- classify the invariant 3D flat structures
1 Nonholonomic Riemannian structures

2 3D simply connected Lie groups

3 Classification of nonholonomic Riemannian structures in 3D
   - Case 1: \( \vartheta = 0 \)
   - Case 2: \( \vartheta > 0 \)

4 Flat structures

5 References
Nonholonomic Riemannian structure \((M, \mathcal{D}, \mathcal{D}^\perp, g)\)

**Ingredients**

- \(M\) is an \(n\)-dim manifold
- \(\mathcal{D}\) is a **completely nonholonomic**, rank \(r < n\) distribution on \(M\)
- \(TM = \mathcal{D} \oplus \mathcal{D}^\perp\), with projectors \(\mathcal{P} : TM \rightarrow \mathcal{D}\) and \(\mathcal{Q} : TM \rightarrow \mathcal{D}^\perp\)
- \(g\) is a fibre metric on \(\mathcal{D}\)

**NH connection** \(\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \rightarrow \Gamma(\mathcal{D})\)

unique connection such that

\[\nabla g \equiv 0 \quad \text{and} \quad \nabla_X Y - \nabla_Y X = \mathcal{P}([X, Y])\]

- parallel transport only along integral curves of \(\mathcal{D}\)
- \(\mathcal{D}\)-curve \(\gamma\) is a **nonholonomic geodesic** if \(\nabla_{\dot{\gamma}} \dot{\gamma} = 0\)
Nonholonomic isometries

**NH-isometry** \((M, D, D^\perp, g) \rightarrow (M', D', D'^\perp, g')\)

diffeomorphism \(\phi : M \rightarrow M'\) such that
\[
\phi_* D = D', \quad \phi_* D^\perp = D'^\perp \quad \text{and} \quad g = \phi^* g'
\]

**Properties**

- preserves the nonholonomic connection: \(\nabla = \phi^* \nabla'\)
- establishes a 1-to-1 correspondence between the nonholonomic geodesics of the two structures
- preserves the projectors: \(\phi_* \mathcal{P}(X) = \mathcal{P}'(\phi_* X)\) for every \(X \in \Gamma(TM)\)

**Left-invariant nonholonomic Riemannian structure** \((M, D, D^\perp, g)\)

- \(M = G\) is a Lie group
- left translations \(L_g : h \mapsto gh\) are NH-isometries
Nonholonomic Riemannian structures in 3D

Contact structure on $M$

We have $\mathcal{D} = \ker \omega$, where $\omega$ is a 1-form on $M$ such that

$$\omega \wedge d\omega \neq 0$$

- fixed up to sign by condition:
  $$|d\omega(Y_1, Y_2)| = 1,$$
  $$(Y_1, Y_2) \text{ o.n. frame for } \mathcal{D}$$

- Reeb vector field $Y_0 \in \Gamma(TM)$:
  $$i_{Y_0}\omega = 1 \quad \text{and} \quad i_{Y_0}d\omega = 0$$

Isometric invariants

- $\vartheta = \|\mathcal{D}(Y_0)\|^2 \geq 0$

- $\kappa, \chi_1, \chi_2$ — curvature invariants

- two natural cases: $\vartheta = 0$ and $\vartheta > 0$
### Bianchi–Behr classification of 3D Lie algebras

#### Unimodular algebras and (simply connected) groups

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>Lie group</th>
<th>Name</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R}^3 )</td>
<td>( \mathbb{R}^3 )</td>
<td>Abelian</td>
<td>Abelian</td>
</tr>
<tr>
<td>( \mathfrak{h}_3 )</td>
<td>( \mathbf{H}_3 )</td>
<td>Heisenberg</td>
<td>nilpotent</td>
</tr>
<tr>
<td>( \mathfrak{se}(1, 1) )</td>
<td>( \text{SE}(1, 1) )</td>
<td>semi-Euclidean</td>
<td>completely solvable</td>
</tr>
<tr>
<td>( \mathfrak{se}(2) )</td>
<td>( \tilde{\text{SE}}(2) )</td>
<td>Euclidean</td>
<td>solvable</td>
</tr>
<tr>
<td>( \mathfrak{sl}(2, \mathbb{R}) )</td>
<td>( \tilde{\text{SL}}(2, \mathbb{R}) )</td>
<td>special linear</td>
<td>semisimple</td>
</tr>
<tr>
<td>( \mathfrak{su}(2) )</td>
<td>( \text{SU}(2) )</td>
<td>special unitary</td>
<td>semisimple</td>
</tr>
</tbody>
</table>

#### Non-unimodular (simply connected) groups

\[
\text{Aff}(\mathbb{R})_0 \times \mathbb{R}, \quad G_{3.2}, \quad G_{3.3}, \quad G^h_{3.4} (h > 0, h \neq 1), \quad G^h_{3.5} (h > 0)
\]
Killing form

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \mathcal{K}(U, V) = \text{tr}[U, [V, \cdot ]]$$

- $\mathcal{K}$ is nondegenerate $\iff$ $\mathfrak{g}$ is semisimple

Completely nonholonomic left-invariant distributions on 3D groups

- no such distributions on $\mathbb{R}^3$ or $G_{3,3}$

Up to Lie group automorphism:

- exactly one distribution on $H_3$, $\text{SE}(1,1)$, $\tilde{\text{SE}}(2)$, $\text{SU}(2)$ and non-unimodular groups
- exactly two distributions on $\tilde{\text{SL}}(2, \mathbb{R})$:
  - denote $\tilde{\text{SL}}(2, \mathbb{R})_{\text{hyp}}$ if $\mathcal{K}$ indefinite on $\mathcal{D}$
  - " denote $\tilde{\text{SL}}(2, \mathbb{R})_{\text{ell}}$ " " definite " "

Barrett, Remsing (Rhodes)  NH Riemannian Structures  CGTA 2017  8 / 18
Case 1: $\mathfrak{V} = 0$

- $D^\perp = \text{span}\{Y_0\}$ determined by $(\mathcal{D}, g)$
- reduces to a sub-Riemannian structure $(M, \mathcal{D}, g)$
- invariant sub-Riemannian structures classified by Agrachev & Barilari (2012)

**Invariants**

- $\{\kappa, \chi_1\}$ form a complete set of invariants for structures on unimodular groups
- structures on non-unimodular groups are further distinguished by discrete invariants
- can rescale structures so that
  
  $$\kappa = \chi_1 = 0 \quad \text{or} \quad \kappa^2 + \chi_1^2 = 1$$
Classification when $\vartheta = 0$
Case 2: $\mathcal{O} > 0$

**Canonical frame $(X_0, X_1, X_2)$**

\[
X_0 = \mathcal{Q}(Y_0) \quad X_1 = \frac{\mathcal{P}(Y_0)}{\|\mathcal{P}(Y_0)\|} \quad X_2 \text{ unique unit vector s.t. } d\omega(X_1, X_2) = 1
\]

- $\mathcal{D} = \text{span}\{X_1, X_2\}$, $\mathcal{D}^\perp = \text{span}\{X_0\}$
- *canonical frame* (up to sign of $X_0$, $X_1$) on $M$

**Commutator relations (determine structure uniquely)**

\[
\begin{align*}
[X_1, X_0] &= c_{10}^{1} X_1 + c_{10}^{2} X_2 \\
[X_2, X_0] &= c_{20}^{0} X_0 + c_{20}^{1} X_1 + c_{20}^{2} X_2 \\
[X_2, X_1] &= X_0 + c_{21}^{1} X_1 + c_{21}^{2} X_2
\end{align*}
\]

\(c_{ij}^k \in C^\infty(M)\)
Left-invariant structures

canonical frame \((X_0, X_1, X_2)\) is left invariant

\(\vartheta, \kappa, \chi_1, \chi_2\) and \(c_{ij}^k\) are constant

NH-isometries preserve the Lie group structure

\((G, D, D^\perp, g)\) NH-isometric to \((G', D', D'^\perp, g')\) w.r.t. \(\phi : G \to G'\) \(\implies \phi = L_{\phi(1)} \circ \phi'\), where \(\phi'\) is a Lie group isomorphism

hence NH-isometries preserve the Killing form \(\mathcal{K}\)

Three new invariants \(\varrho_0, \varrho_1, \varrho_2\)

\[\varrho_i = -\frac{1}{2} \mathcal{K}(X_i, X_i), \quad i = 0, 1, 2\]
Classification

Approach

- rescale frame so that $\vartheta = 1$
- split into cases depending on structure constants
- determine group from commutator relations

Example: $G$ is unimodular and $c_{10}^1 = c_{10}^2 = 0$

\[
[X_1, X_0] = 0 \quad [X_2, X_0] = -X_0 + c_{20}^1 X_1 \quad [X_2, X_1] = X_0 + X_1
\]

implies $K$ is degenerate (i.e., $G$ not semisimple)

(1) $c_{20}^1 + 1 > 0 \implies$ compl. solvable hence on $SE(1, 1)$

(2) $c_{20}^1 + 1 = 0 \implies$ nilpotent " " $H_3$

(3) $c_{20}^1 + 1 < 0 \implies$ solvable " " $\tilde{SE}(2)$

for $SE(1, 1), \tilde{SE}(2): c_{20}^1$ is a parameter (i.e., family of equiv. classes)
Selected results (solvable groups)

\[ \begin{align*}
H_3 & \quad \begin{cases}
[X_1, X_0] = 0 \\
[X_2, X_0] = -X_0 - X_1 \\
[X_2, X_1] = X_0 + X_1
\end{cases} \quad \begin{cases}
\varrho_0 = 0 \\
\varrho_1 = 0 \\
\varrho_2 = 0
\end{cases} \\
SE(1, 1) & \quad \begin{cases}
[X_1, X_0] = -\sqrt{\alpha_1 \alpha_2} X_1 - \alpha_1 X_2 \\
[X_2, X_0] = -X_0 - (1 - \alpha_2) X_1 + \sqrt{\alpha_1 \alpha_2} X_2 \\
[X_2, X_1] = X_0 + X_1
\end{cases} \quad \begin{cases}
\varrho_0 = -\alpha_1 \\
\varrho_1 = -\alpha_2 \\
\varrho_2 = -\alpha_2
\end{cases} \\
\tilde{SE}(2) & \quad \begin{cases}
[X_1, X_0] = -\sqrt{\alpha_1 \alpha_2} X_1 + \alpha_1 X_2 \\
[X_2, X_0] = -X_0 - (1 + \alpha_2) X_1 + \sqrt{\alpha_1 \alpha_2} X_2 \\
[X_2, X_1] = X_0 + X_1
\end{cases} \quad \begin{cases}
\varrho_0 = \alpha_1 \\
\varrho_1 = \alpha_2 \\
\varrho_2 = \alpha_2
\end{cases}
\end{align*} \]

\((\alpha_1, \alpha_2 \geq 0, \alpha_1^2 + \alpha_2^2 \neq 0)\)
Remarks

**Structures on unimodular groups**

- \{\vartheta, \varrho_0, \varrho_1, \varrho_2\} form a complete set of invariants
- \{\vartheta, \kappa, \chi_1\} also suffice for \(H_3, \text{SE}(1, 1), \tilde{\text{SE}}(2)\)
- \(\chi_2 = 0\)

**Structures on 3D non-unimodular groups**

On a fixed non-unimodular Lie group (except for \(G_{3.5}^1\)), there exist at most two non-NH-isometric structures with the same invariants \(\vartheta, \varrho_0, \varrho_1, \varrho_2\)
- exception \(G_{3.5}^1\): infinitely many (\(\varrho_0 = \varrho_1 = \varrho_2 = 0\))
- use \(\kappa, \chi_1\) or \(\chi_2\) to form complete set of invariants
Flat structures

Parallel frames
- an o.n. frame \((X_a)\) for \(\mathcal{D}\) is parallel if \(\nabla X_a \equiv 0\)
- associated to \(\nabla\) is the parallel translation map along a \(\mathcal{D}\)-curve \(\gamma\):
  \[
  \Pi^t_\gamma : \mathcal{D}_\gamma(0) \to \mathcal{D}_\gamma(t)
  \]
- \(\Pi^t_\gamma\) is independent of \(\gamma\) \iff there exists a parallel frame for \(\mathcal{D}\)

\((M, \mathcal{D}, \mathcal{D}^\perp, g)\) is called flat if there exists a parallel frame for \(\mathcal{D}\)

Objective
- classify flat structures in three dimensions
Selected results (unimodular groups)

\( \theta = 0 \)
- all structures are flat

\( \theta > 0 \)
- flat iff. \( \varrho_0 = \theta \varrho_1 \) and \( \varrho_2 = 0 \)
  - \( \varrho_0 < 0 \) \( \implies \) 1-parameter family of structures on \( SE(1, 1) \)
  - \( \varrho_0 = 0 \) \( \implies \) structure on \( H_3 \)
  - \( \varrho_0 > 0 \) \( \implies \) 1-parameter family of structures on \( \widetilde{SE}(2) \)

Interesting observations

- every structure on \( H_3 \) and \( \text{Aff}(\mathbb{R})_0 \times \mathbb{R} \) is flat
- the only flat structures on the semisimple groups are “trivial” structures (those whose NH geodesics are 1-parameter subgroups)
- the remaining groups all admit a 1-parameter family of flat structures (up to equivalence)
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